Quiver Representations over $\mathbb{F}_1$ & Hall Algebras

Changjian Fu

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School of Mathematics, Sichuan University

Presentation Overview

§ 1. Tits Philosophy

§ 2. Quiver Representations over $\mathbb{F}_1$

§ 3. Homological Dimension

§ 4. Hall Algebras

§ 1. Tits Philosophy

History Remarks

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- Tits (1956): incidence geometries $G(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ have combinatorial counterparts that can be interpreted as incidence geometries over $\mathbb{F}_1$.
- Manin (1995): Algebraic geometry over $\mathbb{F}_1$.

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$$\lim_{q\to 1} V(\mathbb{F}_q) \stackrel{?}{=} \{\epsilon_1, \ldots, \epsilon_n\}.$$

What is $\lim_{q\to 1} f$?

Let $V(\mathbb{F}_q)$ and $W(\mathbb{F}_q)$ be finite-dimensional vector spaces over $\mathbb{F}_q$, and $f : V(\mathbb{F}_q) \to W(\mathbb{F}_q)$ an $\mathbb{F}_q$-linear map.

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There exist a basis $\epsilon_1, \dots, \epsilon_n$ of $V(\mathbb{F}_q)$ and a basis $\eta_1, \dots, \eta_m$ of $W(\mathbb{F}_q)$ such that

$$f(\epsilon_1, \dots, \epsilon_n) = (\eta_1, \dots, \eta_m) \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}.$$

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\lim_{q\to 1} f \overset{?}{=}: \quad \lim_{q\to 1} V(\mathbb{F}_q) \quad &\to \lim_{q\to 1} W(\mathbb{F}_q) \\
\epsilon_i \quad &\mapsto \eta_i \quad (i \leqslant r) \\
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One needs 0_V to be an elements in $\lim_{q\to 1} V(\mathbb{F}_q)$!

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- **Linear Maps**: Set maps preserving the base point $*$, which are injective outside the kernel.
- $\mathsf{Vect}(\mathbb{F}_1)$ the category of finite-dimensional vector spaces over $\mathbb{F}_1$.
- **Non-Additive Nature**: $\mathsf{Vect}(\mathbb{F}_1)$ behaves like $\mathsf{Vect}(K)$ except that it is not additive!

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$$\lim_{q\to 1} |Gr_k(\mathbb{F}_q^n)| = \binom{n}{k} = |\mathrm{Gr}_k(\mathbb{F}_1^n)|.$$

§ 2. Quiver Representations over $\mathbb{F}_1$

Defining the Category $\mathrm{Rep}(Q, \mathbb{F}_1)$

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- **Arrow Assignment**:
  For each arrow $\alpha : i \to j$, an $\mathbb{F}_1$-linear map $M_\alpha : M_i \to M_j$.

Morphism $g = (g_i) : M \to N$

$$\begin{tikzcd} M_i \arrow[r, "g_i"] \arrow[d, "M_\alpha"] & N_i \arrow[d, "N_\alpha"] \\ M_j \arrow[r, "g_j"'] & N_j \end{tikzcd}$$

Proto-Exact Structure

Proto-Exact Category

$\mathrm{Rep}_{\mathbb{F}_1}(Q)$ is not an additive category, but a **proto-exact category** based on short exact sequences of pointed sets. $\Rightarrow$ Hall algebra.

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Preservation of Classical Theorems

Despite the highly non-additive framework, classical algebraic properties such as the **Jordan-Hölder Theorem** and **Krull-Schmidt Theorem** still hold perfectly.

Generalization

- All the definitions/results can be naturally extended to the setting of bound quiver with **monomial/commutative** relations.

§ 3. Homological Dimension

Yoneda's Extension

Applying Yoneda's construction, one can define

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$\mathrm{Ext}^i(N, L)$ is not a group but a pointed set with base point

$$L \overset{1}{\rightarrowtail} L \xrightarrow{0} \cdots \xrightarrow{0} N \overset{1}{\twoheadrightarrow} N\,.$$

Global Dimension

The global dimension of $\mathrm{Rep}_{\mathbb{F}_1}(Q)$ is defined as

$$\operatorname{gldim}\mathrm{Rep}_{\mathbb{F}_1}(Q) = \sup\{n \mid \mathrm{Ext}^n(-,-) \neq 0\}.$$

Linear Quivers

Theorem (Fu-Ran-Yang 24)
Let Q be a linear quiver of type $\mathbb{A}_n$.

- *gldim* $\mathrm{Rep}_{\mathbb{F}_1}(Q) = 1$ *iff* $n \leqslant 2$;
- *gldim* $\mathrm{Rep}_{\mathbb{F}_1}(Q) = 2$ *iff* $n \geqslant 3$.

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Keypoint
Indecomposable representations of linear quiver of type $\mathbb{A}_n$ are uniserial!

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- This is not clear for $\mathrm{Ext}^{\geqslant 2}(N, L)$ by definition!
- It is very hard to compute the higher extension spaces!

Infinite-dimensional extension space

Theorem (Fu-Yang-Zeng 25)
Let Q be a cyclic quiver and $\mathrm{Rep}_{\mathbb{F}_1}(Q)_{nil}$ the full subcategory of $\mathrm{Rep}_{\mathbb{F}_1}(Q)$ consisting of nilpotent representations.

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- *gldim* $\mathrm{Rep}_{\mathbb{F}_1}(Q)_{nil} \leqslant 2$;
- *For any nilpotent simple representation S_i and S_j associated with vertices i and j,*

$$\mathrm{Ext}^2(S_i, S_j) \simeq \mathbb{N}.$$

General case: Upper bound

Theorem (Fu-Yang-Zeng 26)
Let Q be an arbitrary quiver (not even finite), then

$$gldim\,\mathrm{Rep}_{\mathbb{F}_1}(Q) \leqslant 2 \qquad and \qquad gldim\,\mathrm{Rep}_{\mathbb{F}_1}(Q)_{nil} \leqslant 2.$$

General case: Upper bound

Keypoint

For any sequence of morphism $L \xrightarrow{f} M \xrightarrow{g} N$ which is exact at M and N, there is a representation W and morphisms fitting into the following commutative diagram

$$\begin{tikzcd} L \arrow[r, tail, "\varphi"] \arrow[d, equal] & W \arrow[r, two heads, "h"] \arrow[d, "\psi"] & N \arrow[d, equal] \\ L \arrow[r, "f"] & M \arrow[r, two heads, "g"] & N. \end{tikzcd}$$

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\end{tikzcd}$$

Compare to the result in derived category $\mathcal{D}^b(kQ)$!

Vanishing of Ext^3

$$\begin{tikzcd}
\epsilon : & L \arrow[r, tail] \arrow[d, equal] & X_1 \arrow[r] & X_2 \arrow[r, "f"] \arrow[d, equal] & X_3 \arrow[r, two heads, "g"] & N \arrow[d, equal] \\
\epsilon_1 : & L \arrow[r, equal] \arrow[d, equal] & L \arrow[u, tail] \arrow[r, "0"] \arrow[d, equal] & X_2 \arrow[r] \arrow[d] & W \arrow[u] \arrow[r] \arrow[d] & N \arrow[d, equal] \\
\epsilon_2 : & L \arrow[r, equal] & L \arrow[r, "0"] & 0 \arrow[r] & N \arrow[r, equal] & N
\end{tikzcd}$$

$$\Rightarrow \epsilon = 0.$$

Classification

Theorem (Fu-Yang-Zeng 26)
Let Q be a finite quiver.

- *gldim* $\mathrm{Rep}_{\mathbb{F}_1}(Q) = 0$ *iff Q consists of a single vertex(type $\mathbb{A}_1$);*
- *gldim* $\mathrm{Rep}_{\mathbb{F}_1}(Q) = 1$ *iff every vertex is either a source or a sink, and is not of type $\mathbb{A}_1$.*

§ 4. Hall Algebras

Hall number

Let (Q, I) be a bound quiver with monomial/commutative relations, and $\mathrm{Rep}_{\mathbb{F}_1}(Q, I)$ the category of $\mathbb{F}_1$-representation over (Q, I). For any $L, M, N \in \mathrm{Rep}_{\mathbb{F}_1}(Q, I)$, the **Hall number** $g_{N,L}^M$ are defined by:

$$g_{N,L}^M = \#\{U \subseteq M \mid U \cong L,\ M/U \cong N\}.$$

Multiplication

The **Hall algebra**

$$\mathcal{H}_{\mathbb{F}_1}(Q,I) = \bigoplus_{M \in \mathrm{Iso}_{\mathbb{F}_1}(Q,I)} \mathbb{Z} u_M$$

is the free $\mathbb{Z}$-module with basis indexed by the isomorphism classes of $\mathrm{Rep}_{\mathbb{F}_1}(Q,I)$, and the multiplication is given by

$$u_N \cdot u_L = \sum_{[M]} g_{N,L}^M u_M.$$

Hall Algebra

Proposition

- ✓ *$\mathcal{H}_{\mathbb{F}_1}(Q, I)$ is an associative algebra with unit u_0.*

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- $\mathcal{H}_{\mathbb{F}_1}(Q,I)$ *is an associative algebra with unit* u_0.
- *Let* $\mathfrak{n}_{\mathbb{F}_1}(Q,I)$ *be the free subgroup generated by indecomposables, then* $\mathfrak{n}_{\mathbb{F}_1}(Q,I)$ *is a Lie subalgebra of* $\mathcal{H}_{\mathbb{F}_1}(Q,I)$.

Comparision

Assume that

- $\mathrm{Rep}_k(Q, I)$ is representation-finite;
- $\mathrm{Rep}_k(Q, I)$ admits Hall polynomials.
- $\mathcal{H}_t(Q, I)$: the generic Hall algebra;
- $\mathcal{H}_1(Q, I)$: its degenerate Hall algebra;
- $\mathfrak{n}(Q, I)$: Lie subalgebra generated by indecomposables.

Comparision

Following Tits philosophy,

- Is there any relationship between $\mathcal{H}_{\mathbb{F}_1}(Q, I)$ and the degenerate Hall algebra $\mathcal{H}_1(Q, I)$?

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Comparision

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- Is there any relationship between $\mathcal{H}_{\mathbb{F}_1}(Q, I)$ and the degenerate Hall algebra $\mathcal{H}_1(Q, I)$?
- Is there any relationship between $\mathfrak{n}_{\mathbb{F}_1}(Q, I)$ and $\mathfrak{n}(Q, I)$?
- Without the existence of Hall polynomials, compare Reidtmann's Lie algebra with $\mathfrak{n}_{\mathbb{F}_1}(Q, I)$?

Nakayma bound quiver

Theorem (Fu-Xiao 26)
Let (Q, I) be a Nakayama bound quiver. There is a bijection between $\mathrm{Iso}_{\mathbb{F}_1}(Q, I)$ and $\mathrm{Iso}_k(Q, I)$. For each triple of representations L, M, N, denote by $\phi^M_{N,L}(t)$ the Hall polynomials.

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- ✓ $\phi^M_{N,L}(1) = g^M_{N,L}$; (*Tits philosophy!*)

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- $\mathcal{H}_1(Q,I) \cong \mathcal{H}_{\mathbb{F}_1}(Q,I)$ *given by* $u_M \mapsto u_M$;

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- $\mathcal{H}_1(Q, I) \cong \mathcal{H}_{\mathbb{F}_1}(Q, I)$ given by $u_M \mapsto u_M$;
- $\mathfrak{n}(Q, I) \cong \mathfrak{n}_{\mathbb{F}_1}(Q, I)$.

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- Similar results hold for gentle-one cylce quiver of $\Lambda(n-1,1,1)$.
- In general, $\mathrm{Rep}_{\mathbb{F}_1}(Q,I)$ may have less representations than $\mathrm{Rep}_k(Q,I)$. One may expect that there is a surjective homomorphism from $\mathcal{H}_1(Q,I)$(resp.$\mathfrak{n}(Q,I)$) to $\mathcal{H}_{\mathbb{F}_1}(Q,I)$ (resp. $\mathfrak{n}_{\mathbb{F}_1}(Q,I)$)!

Thank You!

Questions and Comments are Welcome.