CRiCA I: Reduction of Scattering Diagrams with application to exchange quiver.

Plan.
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§1 Scattering Diagram

$B \in M_n(\mathbb{Z})$: skew-symmetrizable matrix

$S = \mathrm{diag}\{s_1, \cdots, s_n\}$: skew-symmetrizer with $d_i \in \mathbb{Z}_{>0}$

$R = \mathbb{Q}[x_1^{\pm}, \cdots, x_n^{\pm}][[y_1, \dots, y_n]]$ $\quad \mathbb{X}^v = \prod_{i=1}^n x_i^{v_i}$ $\quad \mathbb{Y}^v = \prod_{i=1}^n y_i^{v_i}$

$\forall\ v \in \mathbb{Z}_{\geq 0}^n$. $\quad \mathbb{E}_v : R \longrightarrow R$

$$\mathbb{X}^w \longmapsto \mathbb{X}^w (1+\mathbb{X}^{Bv}\mathbb{Y}^v)^{\frac{v^T S w}{\gcd(Sv)}}$$

$$\mathbb{Y}^{w'} \longmapsto \mathbb{Y}^{w'}$$

ex. $\mathbb{E}_v \in \mathrm{Aut}(R)$. $(\mathbb{E}_v)^{-1}(\mathbb{X}^w) = \mathbb{X}^w(1+\mathbb{X}^{Bv}\mathbb{Y}^v)^{-\frac{v^T S w}{\gcd(Sv)}}$.

$\mathbb{E}_v$ is called formal elementary transformation.

Def. A wall of B in $\mathbb{R}^n$ is a pair (v, W), where

- $0 \neq v \in \mathbb{Z}_{\geq 0}^n$ and $\gcd(v) = 1$
- W is a convex cone spanning $v^\perp := \{m \in \mathbb{R}^n \mid v^T S m = 0\}$.

Rmk. $\{m \in \mathbb{R}^n \mid v^T S m > 0\}$ is called the green side of W.

$\{m \in \mathbb{R}^n \mid v^T S m < 0\}$ is called the red side of W.

Def. ① A Scattering Diagram of B is a collection of (at most countably many) walls of B.

② Let $\mathfrak{D}(B)$ be a SD of B. A smooth path $\rho : [0, 1] \longrightarrow \mathbb{R}^n$ in $\mathfrak{D}(B)$ is finite transverse if

- $\rho(0)$ & $\rho(1)$ are not in any walls of $\mathfrak{D}(B)$
- The image of ρ crosses each wall transversely.
- The image of ρ crosses finitely many walls and does not cross the boundary of walls or intersection of walls which span different hyperplanes.

Def. Let $\mathcal{D}(B)$ be a scattering diagram of B and ρ a finite transverse path crossing walls in $\mathcal{D}(B)$ inorder

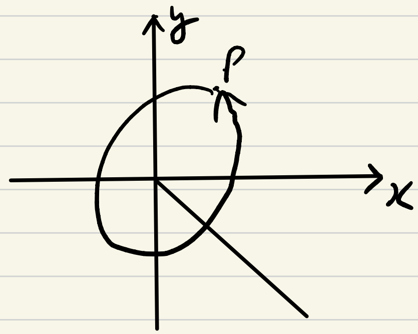
$$(v_1, W_1),\ (v_2, W_2),\ \dots,\ (v_t, W_t)$$

Then define $\overline{\mathbb{E}}_\rho := \overline{\mathbb{E}}_{v_t}^{\varepsilon_t} \cdots \overline{\mathbb{E}}_{v_2}^{\varepsilon_2} \cdot \overline{\mathbb{E}}_{v_1}^{\varepsilon_1} \in \mathrm{Aut}(R)$, which is called the path-ordered product, where $\varepsilon_i = 1$ if ρ crosses W_i from its green side to its red side, $\varepsilon_i = -1$ otherwise.

Def A finite SD is consistent if $\forall$ finite transverse loop ρ. $\overline{\mathbb{E}}_\rho = \mathrm{id}$.

Two finite SDs of B are equivalent if $\forall$ finite transverse path in both SDs determines the same path-ordered product.

e.g. $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \mathbb{R}^2$:



$$\left.\begin{array}{l} (e_1, e_1^\perp) \\ (e_2, e_2^\perp) \\ \left(\begin{pmatrix}1\\1\end{pmatrix}, \mathbb{R}\begin{pmatrix}1\\-1\end{pmatrix}\right) \end{array}\right\} \text{walls of } \mathcal{D}(B)$$

$$\overline{\mathbb{E}}_\rho = \overline{E}_{\binom{0}{1}}^{-1} \overline{E}_{\binom{1}{1}}^{-1} \overline{E}_{\binom{1}{0}}^{-1} \overline{E}_{\binom{0}{1}} \overline{E}_{\binom{1}{0}}$$

$$= \mathrm{id}.$$

§2. Reduction of SD

Let I be a monomial ideal of $\mathbb{Q}[y_1, \cdots, y_n]$.

$\forall\, v \in \mathbb{Z}^n_{\geq 0}$. $\mathbb{E}_v$ induces an automorphism $\mathbb{E}_v \in \mathrm{Aut}(R/I)$.

ex. $\mathbb{E}_v = id \in \mathrm{Aut}(R/I)$ if $y^v \in I$.

Def. A finite SD is consistent mod I, if $\forall$ finite transverse loop ρ,

$$\mathbb{E}_\rho \equiv id \quad \text{in } \mathrm{Aut}(R/I).$$

Two finite SDs of B are equivalent mod I if $\forall$ finite transverse path in both SDs determines the same path-ordered product in $\mathrm{Aut}(R/I)$.

Def. Let I be a monomial ideal of $\mathbb{Q}[y_1, \cdots, y_n]$ and $\mathfrak{D}(B)$ a SD of B.

The reduction $\mathfrak{D}(B)/I$ of $D(B)$ w.r.t I is obtained from $D(B)$ by deleting all walls of the form (v, W) s.t. $y^v \in I$.

Def A SD $\mathfrak{D}(B)$ of B is consistent if $\forall$ monomial ideal I of $\mathbb{Q}[y_1 .. y_n]$ with finite dimensional quotient, the reduction $\mathfrak{D}(B)/I$ is finite and consistent mod I.

Rmk One can define equivalent of SDs.

Thm (GHKK, existence & uniqueness)

$\exists !$ consistent SD $\mathfrak{D}_s(B)$ up to equivalence s.t.

- $(e_i, e_i^\perp)$, $i \in [1, n]$ are walls of $\mathfrak{D}_s(B)$ (called incoming walls)
- For any other wall (v, W), $Bv \notin W$.

Rmk ① GHKK proved a stronger result for any given incoming walls.

② Each connected component of $\mathbb{R}^n \setminus \mathfrak{D}_s(B)$ is a chamber of $\mathfrak{D}_s(B)$.

$\mathbb{R}^n_{>0}$ & $\mathbb{R}^n_{<0}$ are chambers called positive chamber & negative chamber

Def A chamber $\mathcal{C}$ of $\mathfrak{D}_s(B)$ is reachable if $\exists$ a finite transverse path from $\mathbb{R}^n_{>0}$ to $\mathcal{C}$.

Thm (GHKK) Each reachable chamber of $\mathfrak{D}_s(B)$ is of the form

$$\mathbb{R}_{>0} g_1 + \cdots + \mathbb{R}_{>0} g_n,$$

where $G = (g_1, \ldots, g_n)$ is a G-matrix of the cluster alg $\mathcal{A}(B)$.

Muller's reduction Thm:

Let $J = \{j_1 < \cdots < j_p\} \subseteq [1, n]$. Denote

(1) $\pi_J : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $m \longmapsto (m_{j_1}, \cdots, m_{j_p})$

(2) $\pi_J^T : \mathbb{R}^p \longrightarrow \mathbb{R}^n$, $v \longmapsto (0 \cdots 0\, \underset{\uparrow j_1}{v_1}, 0 \cdots 0 \cdots \underset{\uparrow j_p}{v_p}, 0 \cdots 0)$

(3) B_J the principal submatrix of B associated with J.

(4) $\mathcal{D}_0(B_J)$ the SD of B_J in $\mathbb{R}^p$

(5) $\pi_J^*(D_0(B_J)) = \{ (\pi_J^T(V), \pi_J^{-1}(W)) \mid (V, W) \in D_0(B_J) \}$

Thm (Muller) $\pi_J^*(D_0(B_J)) = \mathcal{D}_0(B) / \langle y_j,\ j \notin J \rangle$

Pf: Both $\pi_J^*(D_0(B_J))$ & $\mathcal{D}_0(B)/\langle y_j, j \notin J\rangle$ are consistent SDs in $\mathbb{R}^n$ with incoming walls $\{(e_j, e_j^{\perp}) \mid j \in J\}$. #

§3 Application.

<u>Def</u> The exchange quiver $\vec{\mathcal{H}}(B)$ of $\mathfrak{D}_0(B)$:

Vertex set = Chambers of $\mathfrak{D}_0(B)$

arrow set : $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ if $\exists$ finite transverse path ρ s.t.

- $\rho(0) \in \mathcal{C}_1$, $\rho(1) \in \mathcal{C}_2$
- ρ crosses a unique wall (V, W) from its green side to its red side.

<u>Rmk.</u> The full subquiver of $\vec{\mathcal{H}}(B)$ consisting of reachable chambers is the exchange quiver of $\mathcal{A}(B)$.

<u>Thm</u> [Cao]. $\vec{\mathcal{H}}(B)$ is acyclic. i.e. the exchange quiver of $\mathcal{A}(B)$ is acyclic.

<u>pf:</u> Assume that $\exists$ $\mathcal{C}_1 \xrightarrow[(V_1, W_1)]{} \mathcal{C}_2 \xrightarrow[(V_2, W_2)]{} \cdots \rightarrow \mathcal{C}_t \xrightarrow[(V_t, W_t)]{} \mathcal{C}_1$

$\Rightarrow \exists$ a finite transverse path $\rho: [0, 1] \rightarrow \mathbb{R}^n$ crossing walls (V_1, W_1), (V_2, W_2), $\cdots$, (V_t, W_t) in order.

For any $u=(u_1,\dots,u_n)\in\mathbb{Z}^n$, denote $\deg u=\sum_{i=1}^n u_i$.

Set $k=\min\{\deg v_i+1 \mid 1\le i\le t\}$. Consider $\mathcal{D}_0(B)/I^k$.

Without loss of generality we may assume that $\deg v_1=\cdots=\deg v_t$.

Since $\mathcal{D}_0(B)$ is consistent $\Rightarrow$ $\mathcal{D}_0(B)/I^k$ is consistent mod I^k

$$\Rightarrow \quad \mathbb{E}_p=\mathbb{E}_{v_t}\cdots\mathbb{E}_{v_2}\cdot\mathbb{E}_{v_1}=\mathrm{id} \quad \text{in } \mathrm{Aut}(R/I^k)$$

Let $w\in(\mathbb{Z}_{>0})^n$. $\Rightarrow$ $v_i^TSw>0$. Denote $\dfrac{v_i^TSw}{\gcd(Sv_i)}=b_i>0$

$$\mathbb{E}_{v_1}(\mathbb{X}^w)=\mathbb{X}^w(1+\mathbb{X}^{Bv_1}\mathbb{Y}^{v_1})^{\frac{v_1^TSw}{\gcd(Sv_1)}}$$

$$\equiv \mathbb{X}^w(1+b_1\mathbb{X}^{Bv_1}\mathbb{Y}^{v_1}) \quad \text{mod } I^k$$

$$\mathbb{E}_{v_2}\mathbb{E}_{v_1}(\mathbb{X}^w)\equiv \mathbb{E}_{v_2}(\mathbb{X}^w)(1+b_1\mathbb{E}_{v_2}(\mathbb{X}^{Bv_1})\mathbb{Y}^{v_1})$$

$$\equiv \mathbb{X}^w(1+b_2\mathbb{X}^{Bv_2}\mathbb{Y}^{v_2})(1+b_1\mathbb{X}^{Bv_1}(1+\mathbb{X}^{Bv_2}\mathbb{Y}^{v_2})\mathbb{Y}^{v_1})$$

$$\equiv \mathbb{X}^w(1+b_2\mathbb{X}^{Bv_2}\mathbb{Y}^{v_2})(1+b_1\mathbb{X}^{Bv_1}\mathbb{Y}^{v_1})$$

$$\equiv \mathbb{X}^w(1+b_2\mathbb{X}^{Bv_2}\mathbb{Y}^{v_2}+b_1\mathbb{X}^{Bv_1}\mathbb{Y}^{v_1}) \quad \text{mod } I^k$$

$$\Rightarrow \quad \mathbb{E}_p(\mathbb{X}^w)\equiv\mathbb{X}^w(1+b_t\mathbb{X}^{Bv_t}\mathbb{Y}^{v_t}+\cdots+b_1\mathbb{X}^{Bv_1}\mathbb{Y}^{v_1})\not\equiv\mathbb{X}^w \quad \text{mod } I^k$$

$\square$