On denominator conjecture of cluster algebras

Changjian FU

(Sichuan University)

International Conference on Triangulated Categories and Related Topics

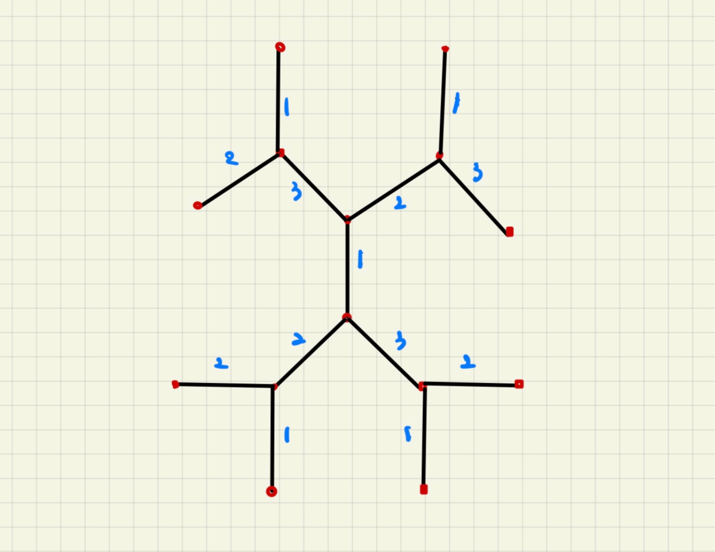
August 11–15, 2024

Outline

Notations

- for an integer a, we write $[a]_+ = \max(a, 0)$;
- n: a positive integer;
- $\mathcal{F}$: the field of rational functions in n indeterminates with coefficients in $\mathbb{Q}$;
- $\mathbb{T}_n$: the n-regular tree whose edges are labeled by the numbers $1, \ldots, n$ such that the n edges emanating from each vertex has different labels.

3-regular tree



Labeled Seed

Definition 1.1

A labeled seed is a pair $(\mathbf{x}, B)$,

- $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set of $\mathcal{F}$;
- $B = (b_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ which is skew-symmetrizable, *i.e.*, there exists a positive integer diagonal matrix S such that SB is skew-symmetric. In this case, S is a skew-symmetrizer of B.

We refer to $\mathbf{x}, x_i, B$ as the cluster, cluster variables and the exchange matrix, respectively.

Seed Mutation

Definition 1.2 (Fomin–Zelevinsky 2002)

Let $(\mathbf{x}, B)$ be a labeled seed and $k \in \{1, \dots, n\}$. The seed mutation μ_k in direction k transforms $(\mathbf{x}, B)$ into $\mu_k(\mathbf{x}, B) := (\mathbf{x}', B')$, where

- the entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

- the cluster variables $\mathbf{x}' = (x'_1, \dots, x'_n)$ are given by

$$x'_j = \begin{cases} \frac{\prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}}{x_k} & \text{if } j = k; \\ x_j & \text{otherwise.} \end{cases}$$

Cluster Pattern

Definition 1.3

Let $(\mathbf{x}, B)$ be a labeled seed. A cluster pattern $t \mapsto \Sigma_t$ of $(\mathbf{x}, B)$ is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, B_t)$ to each vertex t of $\mathbb{T}_n$ such that

- there exists a vertex $t_0 \in \mathbb{T}_n$ such that $\Sigma_{t_0} = (\mathbf{x}, B)$. The vertex t_0 is called a root vertex.
- for an edge $t \overset{k}{\text{———}} t'$ labeled by k of $\mathbb{T}_n$, we have $\Sigma_{t'} = \mu_k(\Sigma_t)$.

We denote by $\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t})$ and $B_t = (b_{ij;t})$.

Cluster Algebra

Definition 1.4 (Fomin–Zelevinsky 2002)

The cluster algebra $\mathcal{A}(B) := \mathcal{A}(\mathbf{x}, B)$ associated with the cluster pattern $t \mapsto \Sigma_t$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by

$$\mathcal{X} = \{x_{i;t}\}_{1 \leq i \leq n, t \in \mathbb{T}_n}.$$

Laurent Phenomenon

Theorem 1 (Fomin–Zelevinsky 2002)

Let $(\mathbf{x}, B)$ be a labeled seed and $t \mapsto \Sigma_t$ a cluster pattern of $(\mathbf{x}, B)$. For any vertices $t, s \in \mathbb{T}_n$ and $1 \leq j \leq n$,

$$x_{j;t} \in \mathbb{Z}[x_{1;s}^{\pm}, \dots, x_{n;s}^{\pm}].$$

Classification Theorem

Definition 1.5

- A cluster algebra is of finite type if there are finitely many cluster variables.
- Let $B = (b_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ be a skew-symmetrizable matrix. The Cartan counterpart of B is a matrix $A(B) = (a_{ij}) \in \mathrm{M}_n(\mathbb{Z})$ such that $a_{ii} = 2, a_{ij} = -|b_{ij}|$ for $i \neq j$.

Theorem 2 (Fomin–Zelevinsky 2003)

Let $\mathcal{A}(B)$ be a cluster algebra with a fixed cluster pattern $t \mapsto \Sigma_t$. The cluster algebra $\mathcal{A}(B)$ is of finite type iff $\exists\ t \in \mathbb{T}_n$ such that the Cartan counterpart of B_t is of finite type generalized Cartan matrix.

$\rightsquigarrow$ We can say a cluster algebra of type $\mathbb{A}, \mathbb{B}, \ldots$.

Example 1.1 (Type $\mathbb{A}_2$)

Let

$$(\mathbf{x} = (x_1, x_2), B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

be a labeled seed. We fix a cluster pattern by assigning $(\mathbf{x}, B)$ to the root vertex t_0:

$$\cdots \overset{1}{\text{———}} t_0 \overset{2}{\text{———}} t_1 \overset{1}{\text{———}} t_2 \overset{2}{\text{———}} \cdots$$

One can compute that

$$\mathcal{A}(B) = \mathbb{Z}[x_1, x_2, \frac{x_1 + 1}{x_2}, \frac{1 + x_1 + x_2}{x_1 x_2}, \frac{1 + x_2}{x_1}].$$

Cluster Monomial

Let $(\mathbf{x}, B)$ be a labeled seed and $t \mapsto \Sigma_t$ a cluster pattern. Fix a vertex s and denote by $\mathbf{x}_s = (x_1, \ldots, x_n)$.

- For each $t \in \mathbb{T}_n$, monomials in the cluster $\mathbf{x}_t$ are cluster monomials.

e.g. $x_{1;t}^{k_1} x_{2;t}^{k_2} \cdots x_{n;t}^{k_n}$, where $k_i \geq 0$.

Denominator Vector

- For each cluster monomial m, by the Laurent phenomenon, we may rewrite m uniquely as

$$m = \frac{f(x_1, \cdots, x_n)}{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}},$$

where $d_1, \ldots, d_n \in \mathbb{Z}$ and $f(x_1, \cdots, x_n) \in \mathbb{Z}[x_1, \cdots, x_n]$ such that $\forall x_i \nmid f(x_1, \cdots, x_n)$.

Definition 2.1

The vector $\mathrm{den}^s(m) = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ is called the denominator vector of m (with respect to the cluster $\mathbf{x}_s$).

Example 2.1 (type $\mathbb{A}_2$)

$$\mathrm{den}^{t_0}(x_1) = (-1, 0), \mathrm{den}^{t_0}(x_2) = (0, -1),$$

$$\mathrm{den}^{t_0}(\frac{x_1+1}{x_2}) = (0, 1), \mathrm{den}^{t_0}(\frac{x_2+1}{x_1}) = (1, 0),$$

$$\mathrm{den}^{t_0}(\frac{x_2+x_1+1}{x_1x_2}) = (1, 1).$$

Note that $\mathbf{x}_{t_1} = (x_1, \frac{x_1+1}{x_2})$. Hence

$$\mathrm{den}^{t_1}(\frac{x_1+1}{x_2}) = (0, -1).$$

$$x_2 = \frac{x_1+1}{\frac{x_1+1}{x_2}} \Rightarrow \mathrm{den}^{t_1}(x_2) = (0, 1).$$

Remark

- Denominator vectors played a key role in the Classification Theorem of cluster algebras of finite type [Fomin–Zelevinsky 2003];
- It has good combinatorial properties studied by [Fomin–Zelevinsky 2003/2007], [Ceballos–Pilaud 2015], [Cao–Li 2020],... via combinatorics.

Denominator Conjecture

$$den : \{\text{Cluster Monomials}\} \longrightarrow \{\text{Denominator Vectors}\} \subseteq \mathbb{Z}^n$$

$$\mathbf{m} \longmapsto \mathrm{den}^{t_0}(\mathbf{m})$$

Conjecture (Fomin–Zelevinsky 2004)

Denominator vectors parametrize cluster monomials, that is, different cluster monomials have different denominator vectors with respect to a given initial cluster.

Known Cases:

- Cluster algebras of rank 2 (*i.e.* $n = 2$) [Sherman–Zelevinsky 2004];
- Cluster algebras of finite type with bipartite initial seeds[Fomin–Zelevinsky 07];
- Acyclic cluster algebras[1] with respect to an acyclic cluster [Caldero–Keller 2006/2008, Rupel–Stella 2020].
- Cluster algebras of rank 3 [Lee–Li–Schiffler 2020].
- Cluster algebras associated with marked surface with particular choice of initial seeds [Fu–Geng, arXiv:2407.11826]

[1] A skew-symmetrizable integer matrix B is acyclic if there does not exist a sequence of indices $i_1, \ldots, i_k$ such that $b_{i_1 i_2} > 0, \ldots, b_{i_{k-1} i_k} > 0, b_{i_k i_1} > 0$. A cluster algebra $\mathcal{A}(B)$ is acyclic if there is a vertex $t \in \mathbb{T}_n$ such that B_t is acyclic. In this case, the cluster $\mathbf{x}_t$ is also called acyclic.

A Weak Form

The Weak Form:

Different cluster variables have different denominator vectors with respect to a given cluster.

The Weak Form is established for:

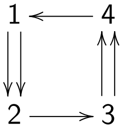
- Cluster algebras of finite type [Geng–Peng 2012, Nakanishi–Stella 2014];
- Cluster algebras of skew-symmetric affine type [Fu–Geng 2019].

Remark

- ⇝ It is not clear how to study this conjecture by combinatorical method in general situation;
- ⇝ We do not have a "correct" categorification for denominator vectors except for certain special cases.

A Counterexample (by Jiarui FEI 2024)

Let Q be the following quiver:



Applying the following mutation sequence $(2, 4, 1, 3, 4, 2, 3, 4)$, one obtains a cluster variable with g-vector $(-3, 2, 0, -1)$. On the other hand, applying the mutation sequence $(4, 2, 3, 1, 2, 4, 1, 2)$, one obtain a cluster variable with g-vector $(0, -1, -3, 2)$. Both cluster variables have denoninator vector $(4, 6, 4, 6)$.

Main Result

Theorem 3 (Fu–Geng working in progress)

Denominator conjecture is true for cluster algebras of finite type.

Type A

Fomin–Shapiro–Thruston Correspondence:

Cluster Algebra type A		Disk with triangulation **T**
initial cluster variables	$\leftrightarrow$	arcs in **T**
non initial cluster variables	$\leftrightarrow$	arcs not in **T**
cluster monomials	$\leftrightarrow$	finite multisets of compatible arcs
denominator vectors	$\leftrightarrow$	intersection vectors

Type A

Lemma 3.1

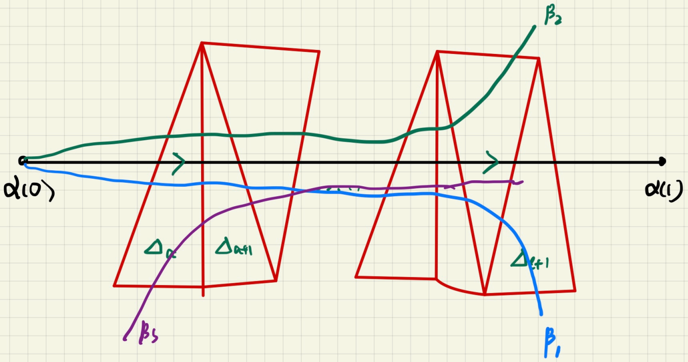
Let $(\mathbb{S}, \mathbb{M})$ be a disk with $n+3$ marked points and $\mathbf{T}$ a triangulation. Let $\mathcal{M}$ and $\mathcal{N}$ be finite multisets consisting of pairwise compatible arcs. If $\mathrm{Int}_{\mathbf{T}}(\mathcal{M}) = \mathrm{Int}_{\mathbf{T}}(\mathcal{N})$, then $\mathcal{M} = \mathcal{N}$.

- Each arc γ not in $\mathbf{T}$ is divided by $\mathbf{T}$ into irreducible arc segments $\gamma^{(1)}, \dots, \gamma^{(m)}$ and each $\gamma^{(i)}$ lies in a triangle Δ_i.
- The equivalence relation on arcs induces an equivalence relation on irreducible arc segments.
- Denote by $\mathrm{arc}\,\mathcal{M}$ the multiset of irreducible arc segments of arcs of $\mathcal{M}$.
- $\mathrm{Int}_{\mathbf{T}}(\mathcal{M}) = \mathrm{Int}_{\mathbf{T}}(\mathcal{N})$ implies $\mathrm{arc}\,\mathcal{M} = \mathrm{arc}\,\mathcal{N}$.

Sketch of Proof

- Assume the contrary, we may assume that $\mathcal{M} \cap \mathcal{N} = \emptyset$.
- ◇ There exists an arc $\alpha \in \mathcal{M}$ (with a fixed orientation) such that all other arcs of $\mathcal{M}$ lie on the left hand side of α.
- By arc $\mathcal{M} =$ arc $\mathcal{N}$, we choose an arc $\beta \in \mathcal{N}$ such that β has maximal common consecutive irreducible arc segments as α.
- By discussion of β, we deduce that there is an arc $\gamma \in \mathcal{M}$ which lies on the right hand side of α, a contradiction.

Sketch of Proof



If $\beta = \beta_1$, then $\exists\ \alpha_1 \in \mathcal{M}$ lies on the right hand side of α $\boxtimes$.

$\Rightarrow \beta = \beta_2$, $\Rightarrow \exists\ \beta_3 \in \mathcal{N}$ lies on the right hand side of α

$\Rightarrow \exists\ \alpha_2 \in \mathcal{M}$ Lies on the right hand side of α. $\boxtimes$.

Remark 4

Lemma 3.1 has been proved in [Fu–Geng, arXiv:2212.11497] in the full generality of tilings, which extends a classical result of [Moshe 1983]: an arc is uniquely determined by its intersection vector for a marked surface with triangulation.

Type B and C

Type C cluster algebras admit geometric models by disk with one unmarked boundary component inside, which can be proved as type A.

Theorem 5 (Fu–Geng 2022, arXiv:2212.11497)

The denominator conjecture is true for a cluster algebra of type C_n if and only if it is true for a cluster algebra of type B_n.

- denominator vector of a non-initial cluster variable equals its f-vector[Gyoda 2021];
- the initial-finial duality of F-matrices [Fujiwara–Gyoda 2019];
- the n denominator vectors of a cluster of a type C_n cluster algebra are linear independent over $\mathbb{Q}$[Fu–Geng–Liu 2021].

Type D

- Geometric model of disk with a puncture;
- Fomin–Shapiro–Thurston's correspondence: denominator vector=intersection vector;
- A similar local-global criterion as tiling.

Exceptional types: An algorithm

Input: A skew-symmetrizable integer matrix B of finite type.

- Compute the set $\mathcal{S}$ of equivalence classes of matrices which can be obtained from B by mutations;
- For each $C \in \mathcal{S}$, compute its D-matrices associated to a cluster pattern of C, say, $\mathcal{D} = \{D_1, \ldots, D_m\}$;
- Compute the determinant $|D_i|$ for each $D_i \in \mathcal{D}$. If $\exists i$, s.t. $|D_i| = 0$, then the DC is false; Otherwise,

- For any pair D_i and D_j of $\mathcal{D}$, by applying permutation of columns, we may assume that D_i and D_j has precise r common columns, which are exactly the first r columns of D_i and D_j. Let $A_{ij} := D_j^{-1} D_i \in M_n(\mathbb{Q})$. Solving the following system of linear inequalities $\begin{pmatrix} A \\ E_n \end{pmatrix} X \geq \begin{pmatrix} 0 \\ e_k \end{pmatrix}$ for each $r < k \leq n$. If $\exists k$, s.t. the system has a solution, then DC is false.

Otherwise, DC is true for the cluster algebra associated with B.

A Consequence

1. $\mathcal{C}$: a cluster category of Dynkin type.
2. T: a basic cluster-tilting object, i.e., $\mathrm{Ext}^1_{\mathcal{C}}(T,T)=0$ and if $\mathrm{Ext}^1_{\mathcal{C}}(T,X)=0$, then $X\in\mathrm{add}\,T$.
3. $A:=\mathrm{End}(T)$: cluster-tilted algebra of Dynkin type.
4. An A-module M is τ-rigid if $\mathrm{Hom}_A(M,\tau M)=0$, where τ is the Auslander-Reiten translation.

Corollary 3.2

Let A be a cluster-tilted algebra of Dynkin type, then different τ-rigid A-module have different dimension vector.

Thanks for your attention!